# EN5101 Digital Control Systems
State Feedback Control

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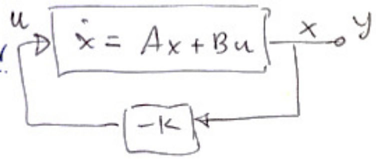
## Pole Placement with State Feedback

- locate closed loop poles at desired locations using feedback principle.

feedback law
$u = -Kx$




$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

assume that all state variables are available for feedback.

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with feedback (state).

$$\dot{x} = Ax + B(-K)x$$

$$sX(s) - x(0) = AX(s) + B(-K)X(s)$$

$$X(s) = (sI - A + BK)^{-1} x(0)$$

$$= \frac{(\text{cof matrix})}{\det[sI - A + BK]} x(0)$$

characteristic eq^n $\det[sI - A + BK] = 0$ — (1)

Assume that the desired char eq^n is as follows

$$(s-d_1)(s-d_2)\cdots(s-d_n) = 0 \quad \text{— (2)}$$

where $d_1 \cdots d_n$ are the desired closed loop poles that provide required transient behavior

Matching coefficients eq. (1) ≡ (2) $k_1, k_2 \cdots k_n$ feedback gains can be calculated.

# Example

state space model of a pendulum with frequency $\omega_0$ is given by

$$\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} = \begin{pmatrix}0&1\\-\omega_0^2&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}u$$

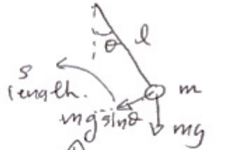
use state feedback to locate the closed loop poles at $-2\omega_0$
[this will double the frequency and set damping ratio to unity].

Sys$^n$ characteristic eq$^n$

$$|sI - A + BK| = 0$$

$$\left|\begin{pmatrix}s&0\\0&s\end{pmatrix}-\begin{pmatrix}0&1\\-\omega_0^2&1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}(k_1\ k_2)\right|=0$$

$$\left|\begin{pmatrix}s&-1\\\omega_0^2&s\end{pmatrix}+\begin{pmatrix}0&0\\k_1&k_2\end{pmatrix}\right|=0$$




$-mg\sin\theta$ acceleration along tangent

$s = l\sin\theta\theta = l\theta$

$\frac{ds}{dt} = l\frac{d\theta}{dt}$

$\frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}$

$l\ddot\theta = -g\theta$

$\underset{\dot x_2}{\ddot\theta} + 0\underset{x_2}{\dot\theta} + \frac{g}{l}\underset{x_1}{\theta} = 0$

$$\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix}=\begin{pmatrix}0&1\\-g/l&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$\begin{vmatrix}s&-1\\\omega_0^2+k_1&s+k_2\end{vmatrix}=0$$

$$s(s+k_2)+(\omega_0^2+k_1)=0$$

$$s^2+k_2 s+(k_1+\omega_0^2)=0 \quad \text{—— (1)}$$

The desired char$^c$ eq$^n$ is

$$(s+2\omega_0)(s+2\omega_0)=0$$

$$s^2+4\omega_0 s+4\omega_0^2=0 \quad \text{—— (2)}$$

(1) = (2)  $k_2 = 4\omega_0$ — (3)

$k_1+\omega_0^2 = 4\omega_0^2$ — (4) $\Rightarrow$ $k_1 = 3\omega_0^2$

Feedback gain vector is $[3\omega_0^2,\ 4\omega_0]$

# Example

Design a state feedback regulator for the system

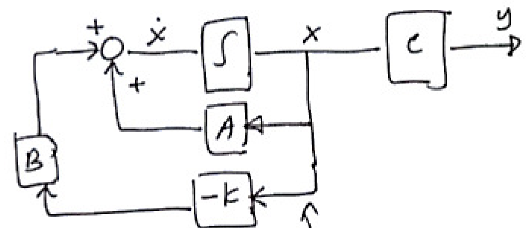
$$\dot X = \overset{A}{\begin{bmatrix}0&1&0\\0&0&1\\-1&-5&-6\end{bmatrix}}X + \overset{B}{\begin{bmatrix}0\\0\\1\end{bmatrix}}u \qquad y = \overset{C}{[1\ 0\ 0]}X$$

and place the closed loop poles at

$-2 \pm j4$ and $-10$

lets use state feedback as

$u = -KX$ ; $K = [k_1\ k_2\ k_3]$ feedback gain vector




need all state variables for feedback

note: in root-locus, Bode methods, we need only the output, not all the states.

closed-loop poles are the roots of char' equation

$$|sI - A + BK| = 0$$

$$\left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1+k_1 & s+k_2 & s+6+k_3 \end{vmatrix} = 0$$

$$s^3 + (6+k_3)s^2 + (5+k_2)s + (1+k_1) = 0 \quad \text{——— (1)}$$

according to the desired close-loop poles, the required char$^{c}$ equation is

$$(s+2+j4)(s+2-j4)(s+10) = 0$$

$$s^3 + 14s^2 + 60s + 200 = 0 \quad \text{——— (2)}$$

By equating coefficients of (1) and (2)

$$6 + k_3 = 14 \Rightarrow k_3 = 8$$
$$5 + k_2 = 60 \Rightarrow k_2 = 55$$
$$1 + k_1 = 200 \Rightarrow k_1 = 199$$

∴ Required feedback gain matrix is

$$K = [199 \quad 55 \quad 8]$$

Matlab

```
A = [0 1 0; 0 0 1; -1 -5 -6];
B = [0; 0; 1];
P = [-2+j*4 -2-j*4 -10];
K = place(A,B,P)
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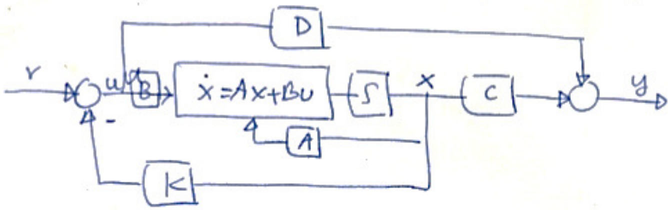
# Error Dynamics of State Feedback

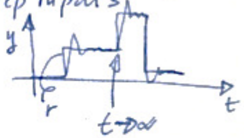



$$\dot{x} = Ax + Bu \quad \text{— (1)}$$
$$y = Cx + Du$$


Control law (state feedback)

$u(t) = r(t) - Kx(t)$ — (2) $\quad r(t) = 0$ for regulator (reject offsets and disturbances)

$\neq 0$ for servo systems (motion control) step inputs




Closed loop system

$$\dot{x}(t) = Ax + B(r - Kx)$$
$$= (A - BK)x + Br \quad \text{— (3)}$$

as $t \to \infty$

$$\dot{x}(\infty) = (A-BK)x(\infty) + Br \quad \text{— (4)}$$

(3) − (4)

$$\dot{x}(t) - \dot{x}(\infty) = (A-BK)(x(t) - x(\infty))$$

$$\dot{e}(t) = (A-BK)e(t) \qquad \text{Error dynamics}$$

Laplace $\quad sE(s) - e(0) = (A-BK)E(s)$

$$E(s) = (sI - A + BK)^{-1} e(0)$$

Design $K$ s.t. $e(t) \to 0$ for any $e(0) \neq 0$

Find $K = [k_1 \cdots k_n]$ to locate closed-loop poles at arbitrary locations